Derivative

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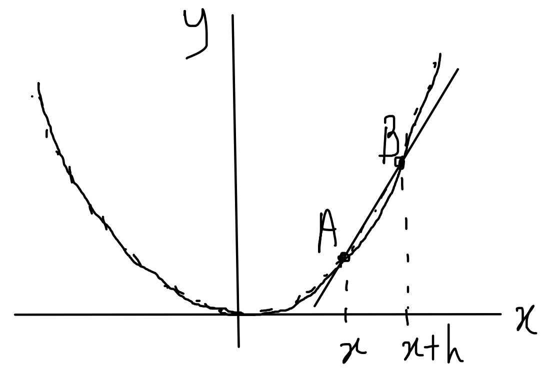
e.g. $y = x^2$

(i) Find gradient of AB.

(ii) Find gradient of tangent at A.



Answer.

(i) At A, $y = x^2$
At B, $y = (x+h)^2$.

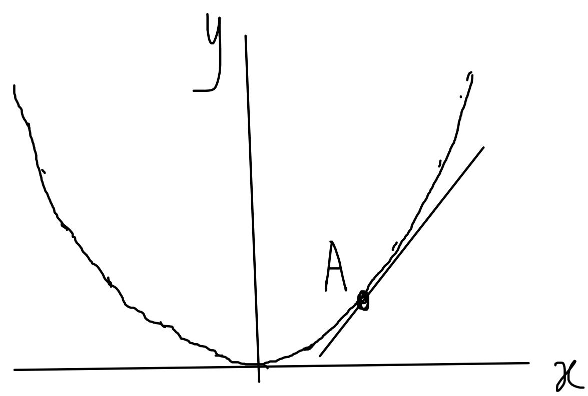
$$\text{Gradient } AB = \frac{(x+h)^2 - x^2}{(x+h) - x}$$

$$= \frac{(x^2 + 2xh + h^2) - x^2}{h}$$

$$= \frac{2xh + h^2}{h} = 2x + h$$

(ii) When $h \to 0$,
AB → tangent at A.

So gradient of tangent at A



$$= 2x + h = 2x + 0 = 2x.$$

Gradient of a tangent to a $y = f(x)$ graph is called a <u>derivative</u>.

Rate of Change

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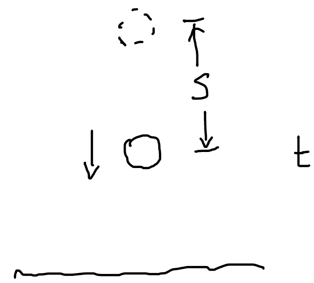
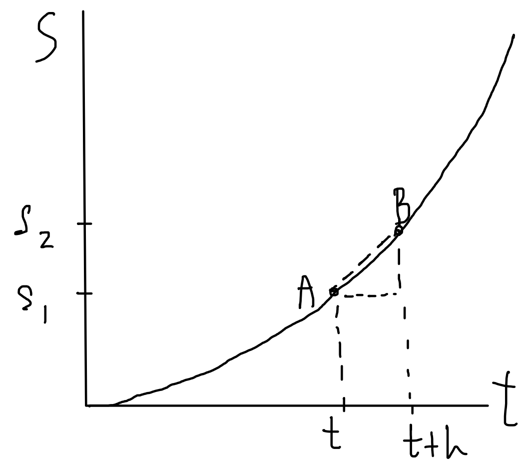
e.g. I dropped a stone.
Before it hits the ground,
the distance it falls is
$s = 5t^2$. t is time in seconds.
s is distance in metres.



Find its speed when $t = 0.5$ s.

Answer:



Let $s_1 = f(t) = 5t^2$

After a short time h,

$s_2 = f(t+h) = 5(t+h)^2$

$$\text{Speed} = \frac{\text{distance travelled}}{\text{time taken}}$$

$$\therefore \text{speed} = \frac{f(t+h) - f(h)}{h} = \text{gradient on graph.}$$

But gradient changes at different t.

To find speed at t, h must be very small $\to 0$.

So must find gradient to tangent at t.

From last page: $s = t^2 \to$ gradient $= 2t$

$s = 5t^2 \to$ gradient $= 10t = 10 \times 0.5 = 5$ m/s

Speed is <u>rate of change</u> of distance.

Use of standard notations $f'(x)$, $f''(x)$, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ $\left[= \frac{d}{dx}\left(\frac{dy}{dx}\right)\right]$

Notations for Derivative

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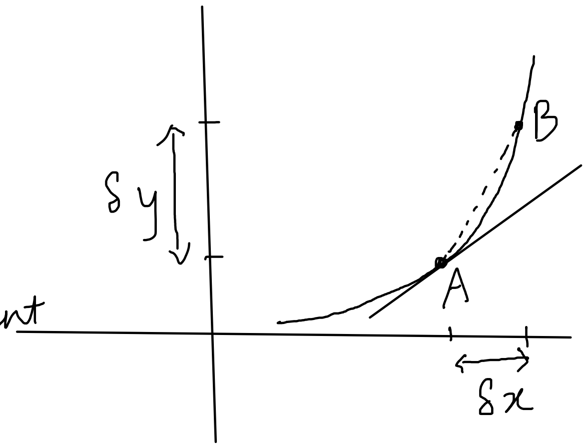
e.g. Gradient of AB $= \frac{\delta y}{\delta x}$

If $\delta x \to 0$, then

$\frac{\delta y}{\delta x} \to$ gradient of tangent at A.



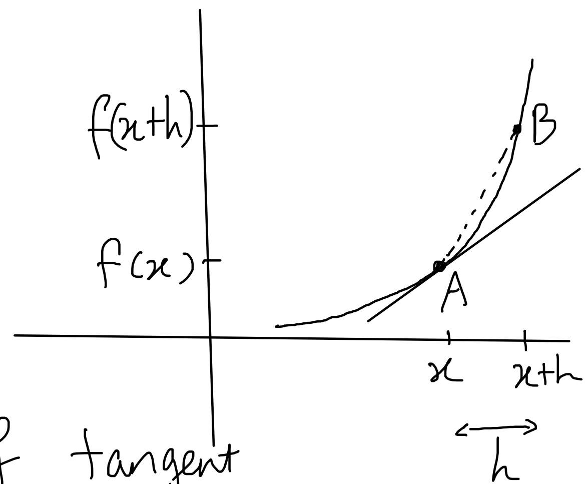
Notation $\boxed{\frac{dy}{dx} = \frac{\delta y}{\delta x} \text{ when } \delta x \to 0}$ (1st derivative)

e.g. gradient of AB

$= \frac{f(x+h) - f(x)}{h}$

If $h \to 0$, then

$\frac{f(x+h) - f(x)}{h} \to$ gradient of tangent at A.



Notation $\boxed{f'(x) = \frac{f(x+h) - f(x)}{h} \quad \text{when } h \to 0}$

So $\frac{dy}{dx} = f'(x)$.

Can also plot graph of $f'(x)$ against x. Its gradient is written as $\boxed{\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = f''(x)}$ (2nd derivative)

Derivatives of x^n, for any rational n, sin x, cos x, tan x, e^x, and ln x, together with constant multiples, sums and differences

Formulae for Derivatives

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y	$\frac{dy}{dx}$	y	$\frac{dy}{dx}$
x^n	nx^{n-1} ....rational no.	$\tan x$	$\sec^2 x$
$\sin x$	$\cos x$	e^x	e^x
$\cos x$	$-\sin x$	$\ln x$	$\frac{1}{x}$

Must use radians for trigonometric functions!

<u>Constant multiple</u> $\quad y = af(x) \rightarrow \frac{dy}{dx} = af'(x)$

e.g. $\quad y = 3\tan x \rightarrow \frac{dy}{dx} = 3\sec^2 x$

$s = 5t^2 \rightarrow \frac{ds}{dt} = 5(2t) = 10t$

<u>Sum</u> $\quad y = f(x) + g(x) \rightarrow \frac{dy}{dx} = f'(x) + g'(x)$

e.g. $\quad y = \sin x + \ln x \rightarrow \frac{dy}{dx} = \cos x + \frac{1}{x}$

$y = x^{-2} + x^{10} \rightarrow \frac{dy}{dx} = -2x^{-3} + 10x^9$

<u>Difference</u> $\quad y = f(x) - g(x) \rightarrow \frac{dy}{dx} = f'(x) - g'(x)$

e.g. $\quad y = x^{\frac{1}{2}} - \cos x \rightarrow \frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} + \sin x$

Products and Quotients

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Derivative of Product

$$y = u(x)\,v(x)$$

$$\boxed{\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}}$$

e.g. $y = x \sin x$

$$\frac{dy}{dx} = x\frac{d}{dx}(\sin x) + (\sin x)\frac{dx}{dx}$$

$$= x\cos x + (\sin x)(1)$$

$$= x\cos x + \sin x$$

Derivative of Quotient

$$y = \frac{u(x)}{v(x)}$$

$$\boxed{\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}}$$

e.g. $y = \dfrac{\ln x}{x}$

$$\frac{dy}{dx} = \frac{x\frac{d}{dx}(\ln x) - (\ln x)\frac{dx}{dx}}{x^2}$$

$$= \frac{x\left(\frac{1}{x}\right) - \ln x}{x^2}$$

$$= \frac{1 - \ln x}{x^2}$$

Composite Functions

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Derivative of $y = u(v(x))$

is $$\boxed{\frac{dy}{dx} = \frac{du}{dv} \cdot \frac{dv}{dx}}$$

e.g. $y = \sin(x^2)$ Let $v = x^2$
So $u = \sin v$

$$\frac{dy}{dx} = \frac{du}{dv} \cdot \frac{dv}{dx} = (\cos v) \cdot (2x)$$
$$= 2x \cdot \cos x^2$$

e.g. $y = (x^2 + 2)^3$ Let $v = x^2 + 2$
So $u = v^3$

$$\frac{dy}{dx} = \frac{du}{dv} \cdot \frac{dv}{dx} = 3v^2 \cdot (2x)$$
$$= 6x(x^2 + 2)^2$$

e.g. $y = e^{x^2}$ Let $v = x^2$
$u = e^v$

$$\frac{dy}{dx} = \frac{du}{dv} \cdot \frac{dv}{dx} = e^v \cdot 2x$$
$$= 2x e^{x^2}$$

Increasing and Decreasing

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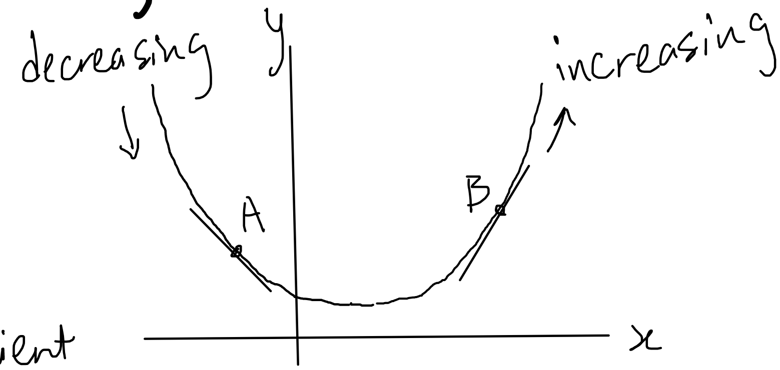
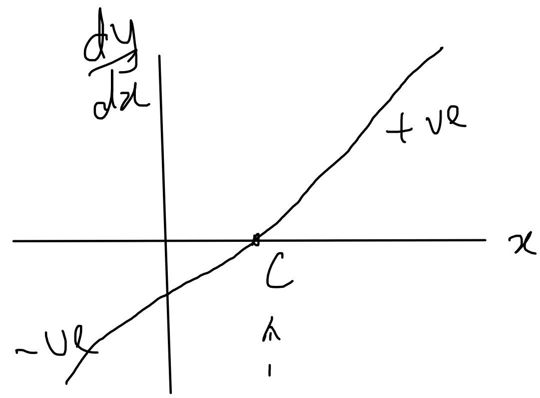
e.g. $y = x^2 + 2x + 2$



At points where function ↓ like A, tangent has $-$ve gradient

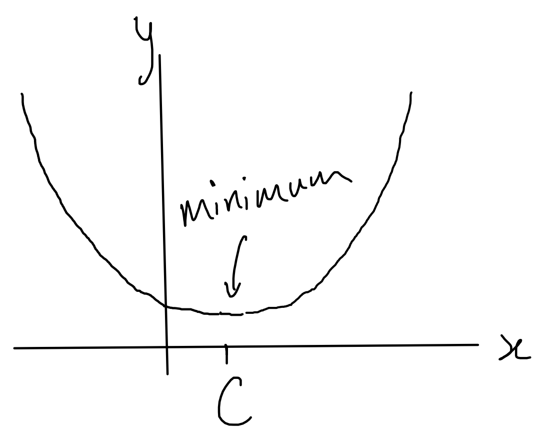
Where function ↑, tangent has +ve gradient.

So a graph of the gradient may look like this:



So in between +ve and $-$ve, the gradient must go thru' zero.

Since zero gradient means horizontal tangent, this must be at the minimum of y:



∴ at the <u>minimum</u> of a function, $\frac{dy}{dx} = 0$.

Stationary Points

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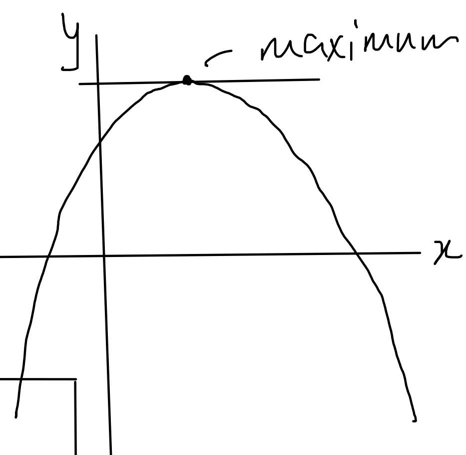
e.g. $y = -x^2 - 2x + 4$



At the maximum of the graph, gradient of tangent is also zero

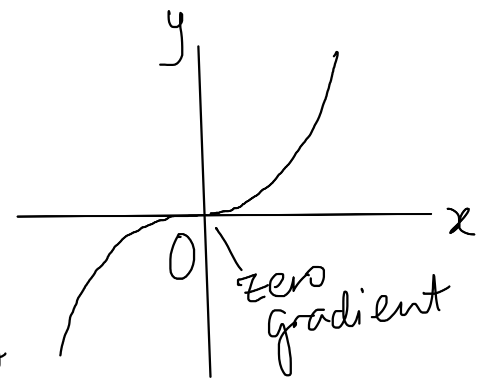
$\therefore$ at the minimum and the maximum of a function, $\dfrac{dy}{dx} = 0$.

But $\dfrac{dy}{dx} = 0$ does not always mean max. or min.

e.g. $y = x^3$

Tangent at $x = 0$ is horizontal.

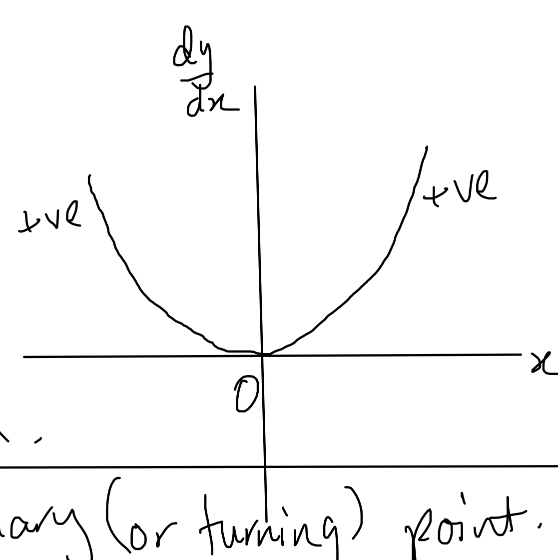
So $\dfrac{dy}{dx} = 0$, but graph is not max. or min.



On both sides, gradient is +ve. But at $x = 0$, gradient $= 0$.

A graph of the gradient

$\rightarrow$ $\dfrac{dy}{dx} = 0$ when gradient is minimum.



Point of $\dfrac{dy}{dx} = 0$ called stationary (or turning) point.

Point of min. (or max.) gradient called inflexion point

Maximum and Minimum

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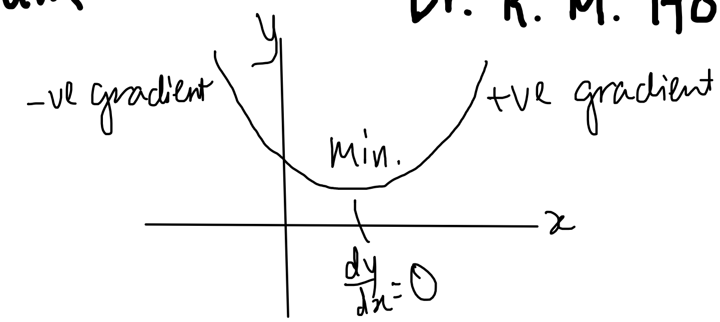
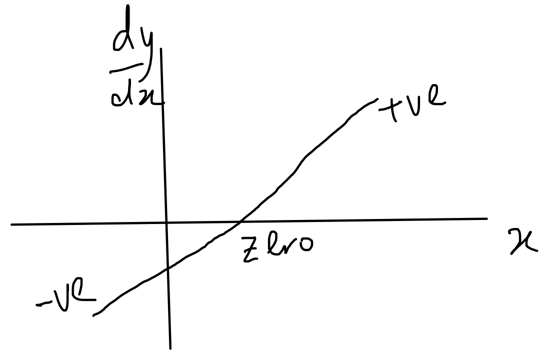
e.g. $y = x^2 + 2x + 2$.

has minimum.



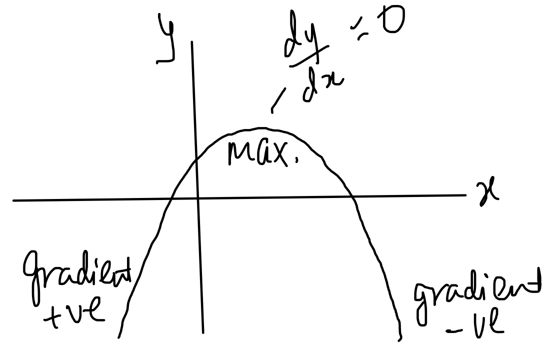
Sketch graph of gradient



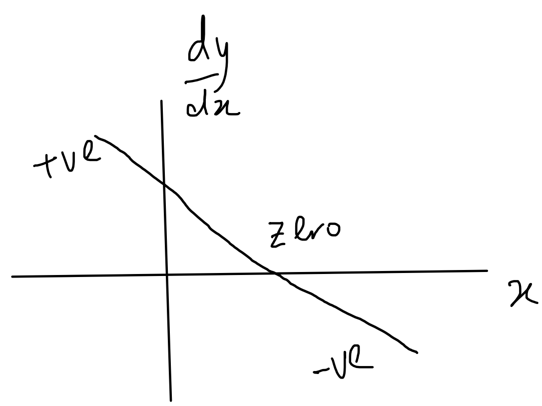
Gradient $\frac{d^2y}{dx^2}$ of this graph is +ve.

e.g. $y = -x^2 - 2x$

has maximum point.



Sketch graph of gradient



Gradient $\frac{d^2y}{dx^2}$ of this graph is -ve.

	Minimum	Maximum	Inflexion
Conditions	$\frac{d^2y}{dx^2} < 0$	$\frac{d^2y}{dx^2} > 0$	$\frac{d^2y}{dx^2} = 0$

Problem 1

Dr. K. M. Hock

2013 P1 Q9 The equation of a curve is $y = 2x^2 + 3x - 5$.

(i) Find the set of values of x for which $y + 3 > 0$.

A particle moves along the curve $y = 2x^2 + 3x - 5$. At the point P the x-coordinate of the particle is increasing at the rate of 0.04 units/sec and the y-coordinate is increasing at 0.2 units/sec.

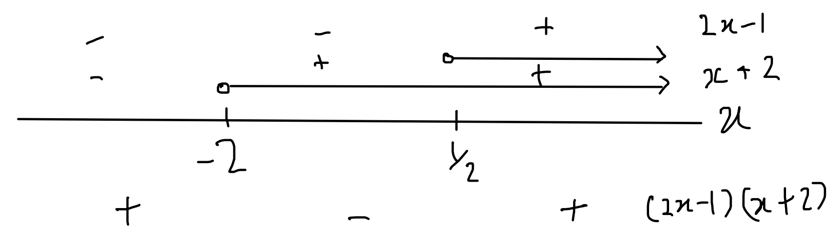
(ii) Find the coordinates of P.

Solution

(i)
$$y + 3 > 0 \quad \Rightarrow \quad 2x^2 + 3x - 5 + 3 > 0$$
$$2x^2 + 3x - 2 > 0$$
$$(2x - 1)(x + 2) > 0$$

$2x - 1 = 0 \Rightarrow x = \frac{1}{2}$

$x + 2 = 0 \Rightarrow x = -2$



$$\therefore \quad x < -2 \quad \text{or} \quad x > \frac{1}{2}$$

(ii) $x = x(t)$, a function of time.

So $y = y(x(t))$, a composite function.

$$\therefore \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

Given $\frac{dx}{dt} = 0.04$ units/sec

$\frac{dy}{dt} = 0.2$ units/sec

$$\frac{dy}{dx} = 4x + 3.$$

So
$$0.2 = (4x + 3)(0.04)$$
$$5 = (4x + 3) \quad \Rightarrow \quad x = \frac{1}{2}$$

$$y = 2x^2 + 3x - 5 = 2\left(\tfrac{1}{2}\right)^2 + 3\left(\tfrac{1}{2}\right) - 5 = \underline{\qquad}.$$

Problem 2

Dr. K. M. Hock

2013 P1 Q11. The equation of a curve is $y = \frac{x^2}{x+2}$.

(i) Obtain expressions for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

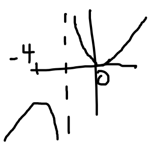
(ii) Determine the nature of each of the stationary points of the curve.

Solution.

(i) Quotient rule $y = \frac{u}{v} \rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Let $u = x^2 \Rightarrow \frac{du}{dx} = 2x$

$v = x+2 \Rightarrow \frac{dv}{dx} = 1$.



$$\therefore \frac{dy}{dx} = \frac{(x+2)(2x) - (x^2)(1)}{(x+2)^2} = \frac{x^2+4x}{(x+2)^2}$$

Quotient rule again:

Let $u = x^2 + 4x$ $\qquad \frac{du}{dx} = 2x+4$

$v = (x+2)^2$ $\qquad \frac{dv}{dx} = 2(x+2)$

$$\frac{d^2y}{dx^2} = \frac{(x+2)^2(2x+4) - (x^2+4x).2(x+2)}{(x+2)^4}$$

$$= \frac{2(x+2)^2 - 2(x^2+4x)}{(x+2)^3} = \frac{8}{(x+2)^3}$$

(ii) $\frac{dy}{dx} = 0 \Rightarrow x^2 + 4x = 0 \Rightarrow x = -4, 0$.

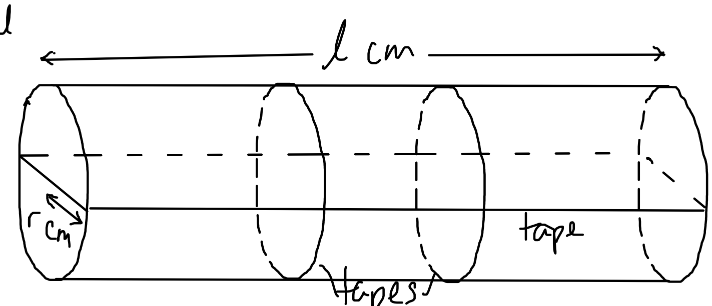
$x = 0$, $\frac{d^2y}{dx^2} = 1$. minimum point.

$x = -4$, $\frac{d^2y}{dx^2} = -1$. maximum point.

Problem 3

Dr. K. M. Hock

2013 P2 Q7. The diagram shows a roll of material held together by 3 tapes. One is in the shape of a rectangle. The other 2 are in circles. Total length of tape is 600 cm.



(i) Show that the volume of the cylinder is $V = \pi r^2(300 - 2r - 2\pi r)$.

(ii) Show that V has a stationary value when $r = \frac{k}{1+\pi}$, where k is a constant. Find k and l.

Solution.

(i) Perimeter of rectangle $= 2r + l + 2r + l = 4r + 2l$

Circumference of circle $= 2\pi r$

Total length of tape
$$600 = (4r + 2l) + 2\pi r + 2\pi r$$
$$300 = 2r + l + 2\pi r$$
$$l = 300 - 2r - 2\pi r$$

Volume $V = \pi r^2 l = \pi r^2 (300 - 2r - 2\pi r)$.

(ii)
$$\frac{dV}{dr} = 600\pi r - 6\pi r^2 - 6\pi^2 r^2 = 0$$
$$100 - r - \pi r = 0$$
$$r = \frac{100}{1+\pi}$$

$\therefore$ $k = 100$.

$$l = 300 - 2\left(\frac{100}{1+\pi}\right) - 2\pi\left(\frac{100}{1+\pi}\right)$$
$$= \frac{300 + 300\pi - 200 - 200\pi}{1+\pi}$$
$$= \frac{100 + 100\pi}{1+\pi} = 100$$